

# THE DISTRIBUTION OF LATTICE POINTS WITH RELATIVELY $r$ -PRIME

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**ABSTRACT.** The distribution of lattice points with relatively  $r$ -prime is related to problems in the Number Theory such as the Extended Lindelöf Hypothesis and the Gauss Circle Problem. It is known that Sitter's result is improved on the assumption of the Extended Lindelöf Hypothesis. In this paper, we improve Sitter's result without assuming the Extended Lindelöf hypothesis.

## 1. INTRODUCTION

The Geometry of Number was used to consider problems of the Number Theory by Minkowski. K. Rogers and H. P. F. Swinnerton-Dyer extended the Geometry of Number over number fields. Let  $K$  be a number field and let  $\mathcal{O}_K$  be its ring of integers. We consider an ordered  $m$ -tuple of ideals  $(\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_m)$  of  $\mathcal{O}_K$  as a lattice point in  $K^m$ . We say that a lattice point  $(\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_m)$  is relatively  $r$ -prime, if there exists no prime ideal  $\mathfrak{p}$  such that  $\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_m \subset \mathfrak{p}^r$ .

In the case  $K = \mathbf{Q}$ , S. J. Benkoski proved that the density of the set of  $m$ -tuples of integers which are relatively  $r$ -prime is  $1/\zeta(rm)$  in 1976 [Be76]. And in general case, B. D. Sitter proved the number of lattice points  $(\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_m)$  in  $K^m$  with relatively  $r$ -prime and  $\mathfrak{N}\mathfrak{a}_i \leq x$  for all  $i = 1, \dots, m$  is

$$\frac{c^m}{\zeta_K(rm)} x^m + (\text{Error term}),$$

where  $\zeta_K$  is the Dedekind zeta function over  $K$  and  $c$  is a constant real number depending only on  $K$  [Si10].

Let  $V_m^r(x, K)$  denote the number of lattice points  $(\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_m)$  with relatively  $r$ -prime and  $\mathfrak{N}\mathfrak{a}_i \leq x$  for all  $i = 1, \dots, m$  and let  $E_m^r(x, K)$  denote its error term, i.e.  $E_m^r(x, K) = V_m^r(x, K) - (cx)^m/\zeta_K(rm)$ .

We considered better upper bound of  $E_m^r(x, K)$  on the assumption of the Extended Lindelöf Hypothesis in [Ta16]. In this paper, we use some classical results to improve Sitter's estimation without assuming the Extended Lindelöf hypothesis. Our main theorem is the following results.

**Theorem.** *Let  $\alpha(n)$  and  $\beta(n)$  be constants*

$$\alpha(n) = \begin{cases} \frac{2}{n} - \frac{8}{n(5n+2)} & \text{if } 3 \leq n \leq 6, \\ \frac{2}{n} - \frac{3}{2n^2} & \text{if } 7 \leq n \leq 9, \\ \frac{3}{n+6} - \varepsilon & \text{if } n \geq 10, \end{cases} \quad \text{and} \quad \beta(n) = \begin{cases} \frac{10}{5n+2} & \text{if } 3 \leq n \leq 6, \\ \frac{2}{n} & \text{if } 7 \leq n \leq 9, \\ 0 & \text{if } n \geq 10. \end{cases}$$

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When  $n = [K : \mathbf{Q}]$ , then we get

$$E_m^r(x, K) = \begin{cases} O(x^{m-\alpha(n)}(\log x)^{\beta(n)}) & \text{if } rm \geq 3, \\ O(x^{2-\alpha(n)}(\log x)^{2\beta(n)+1}) & \text{if } r = 1 \text{ and } m = 2, \\ O(x^{1-\alpha(n)/2}(\log x)^{2\beta(n)}) & \text{if } r = 2 \text{ and } m = 1, \end{cases}$$

for all  $\varepsilon > 0$ .

As a result, we can improve Sittinger's result for  $n \geq 3$ . Moreover, we get better results about  $E_m^r(x, K)$  than assuming the Extended Lindelöf Hypothesis for all number field  $K$  with  $3 = [K : \mathbf{Q}]$ .

## 2. DEDEKIND ZETA FUNCTION OVER $K$

The Dedekind zeta function  $\zeta_K$  over  $K$  is considered as a generalization of the Riemann zeta function and  $\zeta_K$  is defined as

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{\mathfrak{N}\mathfrak{a}^s},$$

with the sum taken over all nonzero ideals of  $\mathcal{O}_K$ .

Lindelöf proposed that for all  $\varepsilon > 0$ , we have  $\zeta(1/2 + it) = O(t^\varepsilon)$  as  $t \rightarrow \infty$  [Li08]. This hypothesis is implied by the Riemann hypothesis, so this is very important hypothesis. The Extended Lindelöf Hypothesis is known as a generalization of the Lindelöf hypothesis. The statement of this Hypothesis is written as follows.

**Extended Lindelöf Hypothesis.** For every  $\varepsilon > 0$ ,

$$\zeta_K\left(\frac{1}{2} + it\right) = O(t^\varepsilon) \text{ as } t \rightarrow \infty.$$

Too many mathematicians tried to get better estimate so far. D. R. Heath-Brown [He88] proved that when  $n = [K : \mathbf{Q}]$

$$(2.1) \quad \zeta_K\left(\frac{1}{2} + it\right) = O(x^{n/6+\varepsilon}) \text{ as } t \rightarrow \infty.$$

This result is known as the best result ever. We used the Extended Lindelöf Hypothesis to consider  $I_K(x)$  in [Ta16]. In this paper, we use one of results obtained from (2.1) instead of the Extended Lindelöf Hypothesis.

## 3. THE NUMBER OF IDEALS

In this section, we prepare for showing the main theorem. Let  $I_K(x)$  be the number of ideals of  $\mathcal{O}_K$  with  $\mathfrak{N}\mathfrak{a} \leq x$ . We consider the value of  $I_K(x)$  to estimate  $E_m^r(x, K)$ , because we know that

$$(3.1) \quad V_m^r(x, K) = \sum_{\mathfrak{N}\mathfrak{a} \leq x^{1/r}} \mu(\mathfrak{a}) I_K\left(\frac{x}{\mathfrak{N}\mathfrak{a}^r}\right)^m.$$

We will consider the sum (3.1), where  $\mu(\mathfrak{a})$  is the Möbius function defined as

$$\mu(\mathfrak{a}) = \begin{cases} 1 & \text{if } \mathfrak{a} = 1, \\ (-1)^s & \text{if } \mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_s, \text{ where } \mathfrak{p}_1, \dots, \mathfrak{p}_s \text{ are distinct prime ideals,} \\ 0 & \text{if } \mathfrak{a} \subset \mathfrak{p}^2 \text{ for some prime ideal } \mathfrak{p}. \end{cases}$$

Considering with this fact, it is important to study the distribution of lattice points, i.e. that of ideals  $I_K(x)$ . In 1993, W. G. Nowak obtained the following result about  $I_K(x)$  [No93].

**Lemma 3.1** (cf. [No93]). *When  $n = [K : \mathbf{Q}]$ , then we get*

$$I_K(x) = cx + \begin{cases} O(x^{1-\frac{2}{n}+\frac{8}{n(5n+2)}}(\log x)^{\frac{10}{5n+2}}) & \text{for } 3 \leq n \leq 6, \\ O(x^{1-\frac{2}{n}+\frac{3}{2n^2}}(\log x)^{\frac{2}{n}}) & \text{for } n \geq 7, \end{cases}$$

where

$$c = \frac{2^{r_1}(2\pi)^{r_2}hR}{w\sqrt{|d_K|}},$$

and:

$h$  is the class number of  $K$ ,

$r_1$  and  $r_2$  is the number of real and complex absolute values of  $K$  respectively,

$R$  is the regulator of  $K$ ,

$w$  is the number of roots of unity in  $\mathcal{O}_K^*$ ,

$d_K$  is discriminant of  $K$ .

For all number field with  $[K : \mathbf{Q}] \geq 10$ , Nowak's result is improved by H. Lao in 2010 [La10]. Following result was shown by using the estimate (2.1).

**Lemma 3.2** (cf. [La10]). *When  $n = [K : \mathbf{Q}]$ , then we get*

$$I_K(x) = cx + O(x^{1-\frac{3}{n+6}+\varepsilon})$$

for every  $\varepsilon > 0$ ,

Combine above lemmas, following lemma about the distribution of ideals holds.

**Lemma 3.3.** *When  $n = [K : \mathbf{Q}]$ , then we get*

$$I_K(x) = cx + \begin{cases} O(x^{1-\frac{2}{n}+\frac{8}{n(5n+2)}}(\log x)^{\frac{10}{5n+2}}) & \text{for } 3 \leq n \leq 6, \\ O(x^{1-\frac{2}{n}+\frac{3}{2n^2}}(\log x)^{\frac{2}{n}}) & \text{for } 7 \leq n \leq 9, \\ O(x^{1-\frac{3}{n+6}+\varepsilon}) & \text{for } n \geq 10, \end{cases}$$

for all  $\varepsilon > 0$

It is too elementary result that the number of lattice points  $(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m)$  in  $\mathbf{Q}^m$  with  $\mathfrak{N}\mathbf{a}_i \leq x$  is  $[x]^m$ . On the other hand, we have no explicit results about the distribution of lattice points in  $K^m$ . By using Lemma 3.3, we can estimate the number of lattice points  $(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m)$  in  $K^m$  with  $\mathfrak{N}\mathbf{a}_i \leq x$ .

#### 4. THE PROOF OF THE MAIN THEOREM

In this section we will show the main theorem. Using the equation (3.1) and Lemma 3.3, an approximation formula for  $E_m^r(x, K)$  is obtained.

**Theorem 4.1.** *Let  $\alpha(n)$  and  $\beta(n)$  be constants*

$$\alpha(n) = \begin{cases} \frac{2}{n} - \frac{8}{n(5n+2)} & \text{if } 3 \leq n \leq 6, \\ \frac{2}{n} - \frac{3}{2n^2} & \text{if } 7 \leq n \leq 9, \\ \frac{3}{n+6} - \varepsilon & \text{if } n \geq 10, \end{cases} \quad \text{and} \quad \beta(n) = \begin{cases} \frac{10}{5n+2} & \text{if } 3 \leq n \leq 6, \\ \frac{2}{n} & \text{if } 7 \leq n \leq 9, \\ 0 & \text{if } n \geq 10. \end{cases}$$

When  $n = [K : \mathbf{Q}]$ , then we get

$$E_m^r(x, K) = \begin{cases} O(x^{m-\alpha(n)}(\log x)^{\beta(n)}) & \text{if } rm \geq 3, \\ O(x^{2-\alpha(n)}(\log x)^{2\beta(n)+1}) & \text{if } r = 1 \text{ and } m = 2, \\ O(x^{1-\alpha(n)/2}(\log x)^{2\beta(n)}) & \text{if } r = 2 \text{ and } m = 1, \end{cases}$$

for all  $\varepsilon > 0$ .

*Proof.* As we remarked above, we know the equation (3.1);

$$V_m^r(x, K) = \sum_{\mathfrak{N}\mathfrak{a} \leq x^{1/r}} \mu(\mathfrak{a}) I_K \left( \frac{x}{\mathfrak{N}\mathfrak{a}^r} \right)^m.$$

First we use Lemma 3.3 and the binomial theorem. Then we obtain

$$\begin{aligned} V_m^r(x, K) &= \sum_{\mathfrak{N}\mathfrak{a} \leq x^{1/r}} \mu(\mathfrak{a}) \left( \frac{cx}{\mathfrak{N}\mathfrak{a}^r} + O \left( \left( \frac{x}{\mathfrak{N}\mathfrak{a}^r} \right)^{1-\alpha(n)} (\log x / \mathfrak{N}\mathfrak{a}^r)^{\beta(n)} \right) \right)^m \\ &= (cx)^m \sum_{\mathfrak{N}\mathfrak{a} \leq x^{1/r}} \frac{\mu(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}^{rm}} + O \left( \sum_{\mathfrak{N}\mathfrak{a} \leq x^{1/r}} \left( \frac{x}{\mathfrak{N}\mathfrak{a}^r} \right)^{m-\alpha(n)} (\log x)^{\beta(n)} \right). \end{aligned}$$

By using the fact  $\sum_{\mathfrak{a}} \frac{\mu(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}^{rm}} = \frac{1}{\zeta_K(rm)}$ , we get

$$V_m^r(x, K) = \frac{c^m}{\zeta_K(rm)} x^m - (cx)^m \sum_{\mathfrak{N}\mathfrak{a} > x^{1/r}} \frac{\mu(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}^{rm}} + O \left( \sum_{\mathfrak{N}\mathfrak{a} \leq x^{1/r}} \left( \frac{x}{\mathfrak{N}\mathfrak{a}^r} \right)^{m-\alpha(n)} (\log x)^{\beta(n)} \right).$$

The first term  $(cx)^m / \zeta_K(rm)$  is known as the principal term of  $V_m^r(x, K)$ , so  $E_m^r(x, K)$  is left terms. Thus

$$E_m^r(x, K) = O \left( x^m \sum_{\mathfrak{N}\mathfrak{a} > x^{1/r}} \frac{\mu(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}^{rm}} + \sum_{\mathfrak{N}\mathfrak{a} \leq x^{1/r}} \left( \frac{x}{\mathfrak{N}\mathfrak{a}^r} \right)^{m-\alpha(n)} (\log x)^{\beta(n)} \right).$$

Now we estimate how fast above first sum grows. From Theorem 3.3 we can estimate  $I_K(x) - I_K(x-1) = O(x^{1-\alpha(n)} (\log x)^{\beta(n)})$ , so we have

$$\begin{aligned} \sum_{\mathfrak{N}\mathfrak{a} > x^{1/r}} \frac{\mu(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}^{rm}} &= O \left( x^m \int_{x^{1/r}}^{\infty} \frac{y^{1-\alpha(n)} (\log y)^{\beta(n)}}{y^{rm}} dy \right) \\ &= O(x^{(2-\alpha(n))/r} (\log x)^{\beta(n)}). \end{aligned}$$

Next we estimate how fast above second sum grows. As well as first sum,  $I_K(x) - I_K(x-1) = O(x^{1-\alpha(n)} (\log x)^{\beta(n)})$  holds, so we have

$$\begin{aligned} \sum_{\mathfrak{N}\mathfrak{a} \leq x^{1/r}} \left( \frac{x}{\mathfrak{N}\mathfrak{a}^r} \right)^{m-1/2+\varepsilon} &= O \left( x^{m-\alpha(n)} \left( 1 + \int_1^{x^{1/r}} \frac{y^{1-\alpha(n)} (\log y)^{\beta(n)}}{y^{r(m-\alpha(n))}} (\log y)^{\beta(n)} dy \right) \right) \\ &= \begin{cases} O(x^{m-\alpha(n)} (\log x)^{\beta(n)}) & \text{if } rm \geq 3, \\ O(x^{2-\alpha(n)} (\log x)^{2\beta(n)+1}) & \text{if } r = 1 \text{ and } m = 2, \\ O(x^{1-\alpha(n)/2} (\log x)^{2\beta(n)}) & \text{if } r = 2 \text{ and } m = 1. \end{cases} \end{aligned}$$

Hence we get

$$E_m^r(x, K) = \begin{cases} O(x^{m-\alpha(n)} (\log x)^{\beta(n)}) & \text{if } rm \geq 3, \\ O(x^{2-\alpha(n)} (\log x)^{2\beta(n)+1}) & \text{if } r = 1 \text{ and } m = 2, \\ O(x^{1-\alpha(n)/2} (\log x)^{2\beta(n)}) & \text{if } r = 2 \text{ and } m = 1. \end{cases}$$

This proves the main Theorem. □

In 2010, B. D. Sittinger showed the following theorem about lattice points with relatively  $r$ -prime over number field  $K$ .

**Theorem 4.2** (cf. [Si10]). *When  $n = [K : \mathbf{Q}]$*

$$E_m^r(x, K) = \begin{cases} O(x^{m-1/n}) & \text{if } m \geq 3, \text{ or } m = 2 \text{ and } r \geq 2, \\ O(x^{2-1/n} \log x) & \text{if } m = 2 \text{ and } r = 1, \\ O(x^{1-1/n} \log x) & \text{if } m = 1 \text{ and } \frac{n(r-2)}{r-1} = 1, \\ O(x^{1-1/n}) & \text{if } m = 1 \text{ and } \frac{n(r-2)}{r-1} > 1, \\ O(x^{(2-1/n)/r}) & \text{if } m = 1 \text{ and } \frac{n(r-2)}{r-1} < 1. \end{cases}$$

Considering the Sittinger's result [Si10], we can improve the order of  $E_m(x, K)$  for all number field  $K$  with  $[K : \mathbf{Q}] \geq 3$ .

## 5. APPENDIX

In this paper we considered about all number fields, but our results can be improved for all abelian extension  $K/\mathbf{Q}$ . It is well known that for an abelian extension  $K/\mathbf{Q}$  with  $[K : \mathbf{Q}] \geq 4$ ,

$$(5.1) \quad I_K(x) = cx + O(x^{1-3/(n+2)}).$$

Using this approximation (5.1), the following better result for abelian extension field  $K$  is obtained in a way similar to the proof of Theorem 4.1.

**Theorem 5.1.** *For all abelian extension field  $K$  with  $[K : \mathbf{Q}] \geq 4$ , we get*

$$E_m^r(x, K) = \begin{cases} O(x^{1-\frac{3}{2(n+2)}+\varepsilon}) & \text{if } r = 2 \text{ and } m = 1, \\ O(x^{m-\frac{3}{n+2}+\varepsilon}) & \text{otherwise.} \end{cases}$$

It goes without saying that the better approximation formula for  $I_K(x)$  we have, the better results for  $E_m^r(x, K)$  we get. On the contrary, if we can get the exact order of  $E_m^r(x, K)$ , the explicit approximation formula for  $I_K(x)$  is obtained from considering the exact order of  $E_m^r(x, K)$ . And it is proven that there are some relation between the distribution of lattice points with relatively  $r$ -prime and problems in the Number Theory in [Ta16]. We think it is important that studying the exact order of  $E_m^r(x, K)$  without using an approximation formula for  $I_K(x)$ .

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